Technical trip report

The purpose of my visit at Space and Naval Warfare System Center (SPAWAR) of San Diego supported by ONRIFO (grant #N00014-02-1-4010) was to meet Dr. S.S. Sritharan in order to continue our joint research on the controllability for the Navier–Stokes and magnetohydrodynamic (MHD) equations and other related problems.

This research started during a previous visit at SPAWAR when we focused on some problems concerning the MHD equations. These equations describe the motion of a viscous incompressible conducting fluid in a magnetic field and consist of a subtle coupling of the Navier–Stokes equations of viscous incompressible fluid flow and the Maxwell equations of electromagnetic field. In a first research project, we concerned ourselves with understanding the role played by the internal hydrodynamic and magnetic forces in creating the dynamics of conducting fluids in magnetic fields. To this aim, we decoupled the Navier–Stokes and Maxwell parts of the MHD equations on small time interval in order to separate the hydrodynamic and magnetic effects. In this way we obtained a splitting approximation scheme whose convergence represented the objective of our joint paper entitled "*Fluid–magnetic splitting of the magnetohydrodynamic equations*" which will appear in "Theoretical and Computational Fluid Dynamics". The second theme we dealt with (before my last visit) was the controllability for the MHD equations. The controlled MHD equation (with boundary and initial conditions) we considered are the following:

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(1)\qquad
\begin{aligned}
&\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y + \nabla p + \nabla\left(\frac{1}{2}B^2\right) && \\
&\qquad\qquad\qquad\quad -(B\cdot\nabla)B = f + \chi_\omega u && \text{in } \Omega\times(0,T),\\
&\frac{\partial B}{\partial t} + \eta\,\mathrm{curl}(\mathrm{curl}\,B) + (y\cdot\nabla)B && \\
&\qquad\qquad\qquad\qquad -(B\cdot\nabla)y = \chi_\omega v && \text{in } \Omega\times(0,T),\\
&\mathrm{div}\,y = 0,\ \mathrm{div}\,B = 0 && \text{in } \Omega\times(0,T),\\
&y = 0,\ B\cdot N = 0,\ (\mathrm{curl}\,B)\times N = 0 && \text{on } \partial\Omega\times(0,T),\\
&y(\cdot,0) = y_0,\ B(\cdot,0) = B_0 && \text{in } \Omega,
\end{aligned}
$$

where Ω is a bounded open set of $\mathrm{I\!R}^3$, ω is an open subset of Ω and $T > 0$ is a fixed time. Besides, $y = (y_1, y_2, y_3) : \Omega \times [0,T] \to \mathrm{I\!R}^3$ is the velocity vector field, $p : \Omega \times [0,T] \to \mathrm{I\!R}$ is the (scalar) pressure and $B = (B_1, B_2, B_3) : \Omega \times [0,T] \to \mathrm{I\!R}^3$ is the magnetic field. The vector functions $u = (u_1, u_2, u_3) : \Omega \times [0,T] \to \mathrm{I\!R}^3$ and $v = (v_1, v_2, v_3 : \Omega \times [0,T] \to \mathrm{I\!R}^3$ are controls distributed in ω, and χ_ω is the characteristic function of ω. Roughly speaking, the controllability result we obtained (together with V. Barbu and T. Havârneanu) amounts to saying that the steady-state (stationary) solutions of the MHD equations are locally controllable provided that they are sufficiently smooth. That is, for such a steady-state solution, if the initial data y_0, B_0 are sufficiently smooth and "close" to this solution, then there exist locally distributed (internal) controls u and v such that the corresponding solutions of the MHD equations starting from these initial data reach the steady-state solution in a fixed finite time. For the Navier–Stokes and Boussinesq equations this result was previously established by O.Yu. Imanuvilov and A.V. Fursikov. The proof is based on a fixed point argument (previously used in the controllability of the Navier–Stokes equations) which reduces our nonlinear situation to a linear one. The main ingredient in proving the global controlability of the null solution of a linear version of (1) consists of two Carleman type inequalities for the Stokes and dynamo equations (estimating their solutions in the entire domain by means of the restrictions of these solutions on the subdomain on which the controls are distributed). This was the stage of our joint research before my last appointment with Dr. S.S. Sritharan.

The first objective we pursued during my last visit at SPAWAR (supported by ONRIFO) was the study of the boundary controllability of the steady-state solutions of the MHD equations. In this case the controls u and v are distributed on a part σ of the boundary (and not in an interior subdomain ω of Ω as in the preceding case). The boundary controllability result we established is the following: *For a given sufficiently smooth steady-state solution of the MHD equations, if the initial data are sufficiently smooth and "close" to this solution, then there exist controls u and v distributed on a given part σ of the boundary $\partial\Omega$ and corresponding solutions of the MHD equations starting from these initial data and reaching the steady-state solution in a fixed finite time.* The idea of the proof is to reduce the boundary controllability problem to the internal controllability one (described before) by a suitable extension of the initial data and the steady-state solution to a larger domain. Besides the boundary controllability for the 3-dimensional MHD equations, we established that internal and boundary controllability results for the 2-dimensional MHD equations can be obtained by using the same approaches as in the 3-dimensional case.

The first equations in (1) without the terms containing B and the equation $\operatorname{div} y = 0$ (in Ω) form a controlled variant of the Navier–Stokes equations. Another objective we proposed was to establish the controllability of the steady-state solutions of the Navier–Stokes equations but with other significant boundary conditions (different from the no-slip boundary condition satisfied by y in (1)). Besides, we concerned ourselves with obtaining the controllability for both Navier–Stokes and MHD equations for more general domains. (Some technical aspects forced us and other authors to restrict the considerations to some special domains.) To this aim, we carefully studied various ways to derive the needed Carleman inequality for the pressure (because this particular inequality requires additional hypotheses on the shape of domain). Then, we advanced in the study of the controllability for the Navier–Stokes with one of the two following slip boundary conditions:

$$y \cdot N = 0, \ \sum_{i,j=1}^{3} \sigma_{ij} N_j T_i = 0 \text{ on } \partial\Omega \text{ for all } T = (T_1, T_2, T_3), \tag{2}$$

$$\sum_{j=1}^{3} \sigma_{ij} N_j = 0 \text{ on } \partial\Omega, \ i = 1, 2, 3, \tag{3}$$

where $N = (N_1, N_2, N_3)$ is the exterior normal to the boundary, $T = (T_1, T_2, T_3)$ is any tangent vector and

$$\sigma_{ij} = \left(\frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right) - p\delta_{ij}.$$

The boundary condition (3) corresponds to a free boundary (contact with vacuum or with a region of given pressure). The boundary condition (2) could correspond to the case when the outer region is filled by an inviscid fluid with much greater density than the one filling Ω. Much of our attention was payed to the contollability for the Navier–Stokes equations in domains with free boundary.

We also discussed about Carleman type inequalities for the linearized vorticity equations in connection with the observability problem for the vorticity equations (obtained by applying "curl" operator to the Navier–Stokes equations).

Finally, we considered a hyperbolic version of the MHD equations (in fact, a parabolic–hyperbolic coupling of the Navier–Stokes and Maxwell equations) arising in the mathematical description of the ionosphere. We drew up a project of studying these equations (the mathematical setting and analysis, controllability etc.).

The research problems we studied were motivated by the control theoretic challenges in variational data assimilation of METOC (meteorology–oceanography) and also in space weather. This visit we paid particular attention to the mathematically difficult interface (free boundary) problems which arise in ocean–atmospheric, atmospheric–ionospheric and ionospheric–magnetospheric coupling assimilations. Coupled assimilation is at the forefront of earth and space weather prediction in DoD these days. Another related technology area that would potentially benefit by our research is active heating of the ionospheric auroral regions. Example of such projects includes the well known HAARP (high frequency active auroral research program) at Alaska. Here the challenge is to heat the ionospheric electrojet with high frequency electromagnetic waves to generate extremely low frequency waves.

April 30, 2002 Prof. Cătălin Popa